ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

TRIMESTER 1 2020/2021

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2. Lines and Their Equations

2.1. Slope of a Line and its y-intercept.

In Euclidean geometry (where there are no coordinates), there are several ways to describe a straight line. For example, we could specify two points that lie on it, or we could specify one point on the line and also insist that the line is parallel to a given line. This is illustrated in Figure 1.

We can either say that Line 2 is the line through the points A and B or we can say that Line 2 is the line through the point A (or B) parallel to to the line 1. In Euclidean geometry, both of these methods are pretty equal in terms of convenience.

However, if we are studying coordinate geometry, then specifying two points makes a lot of calculations more difficult than if we know one point together with the direction of the line. So we will first look at how to define a line using this latter method. Let us first look at the example shown in Figure 2.

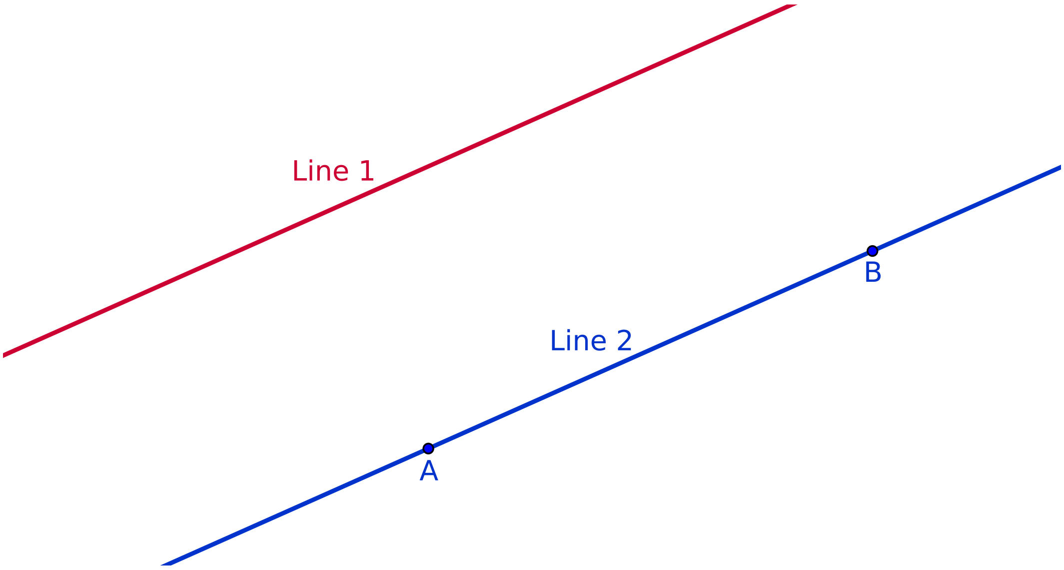


FIGURE 1. Describing a straight line in Euclidean geometry.

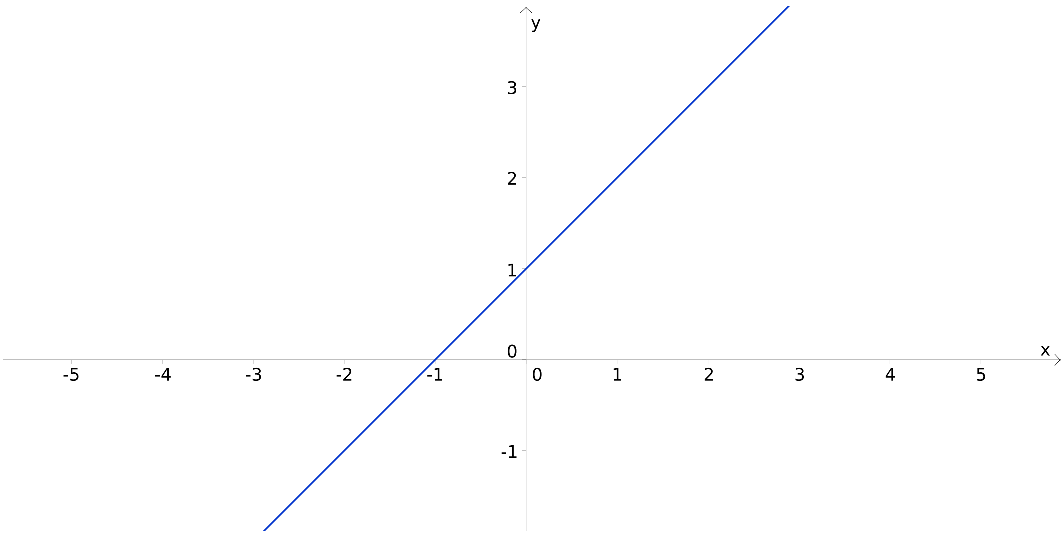


FIGURE 2. Describing a straight line in coordinate geometry.

We want to describe the line using a point on the line and the direction of the line. But the question is which point should we choose and how do we define the direction in terms of a number. Figure 3 answers these questions.

The point we take is the point where the line crosses the y-axis, that is $(0, 1)$. The direction of the line is defined in terms of the slope of the line. The slope is given by $\frac{\text{Rise}}{\text{Run}} = \frac{4-1}{3-0} = 1$. These two pieces of information allow us to write down the equation of the line. The general equation of a line is given by

$$y = (\text{Slope}) \times x + (y\text{-coordinate of } y\text{-intercept}). \tag{1}$$

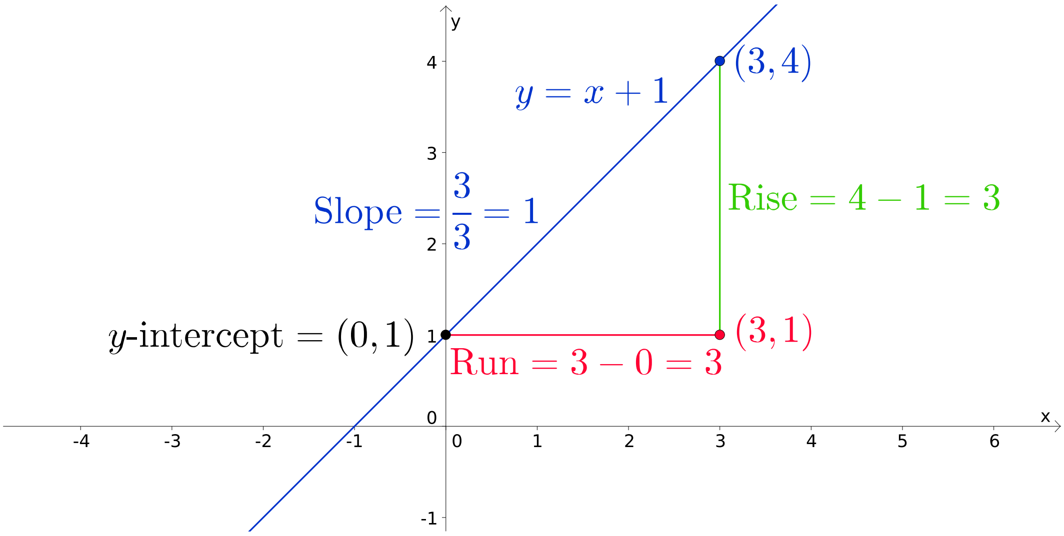


FIGURE 3. Describing a straight line using the slope and the y-intercept.

So in this case the equation is $y = x + 1$, as indicated in Figures 2 and 3. We usually write (1) as $y = mx + c$, where m is the slope and c is the y-coordinate of the point where the line crosses the y-axis. Note that sometimes we will just say c is the y-intercept.

Remark 2.1.1. This description of a line will work for any line except for vertical lines. There are a couple of problems with vertical lines. Firstly the Run is zero, so $\frac{\text{Rise}}{\text{Run}}$ (which is the slope in all other cases) is undefined. Also vertical lines do not intercept the y-axis at one point. A vertical line either is the y-axis or it doesn't touch it at all. This is no great problem though, we define a vertical line by the point where it intercepts the x-axis. So a vertical line through the point $(c, 0)$ has the equation $x = c$.

In the example above, we already had the line plotted on a graph and we wanted to find its equation. More often we will want to find the equation of a line given two points that lie on it. We will now do an example that shows us how to do this, since this will also help us do examples of the first sort, since what we really did above was to find the slope from the points $(0, 1)$ and $(3, 4)$.

Example 2.1.2. Find the equation of the line through the points $(1, 3)$ and $(5, 5)$. We first have to find the slope of the line.

Looking at Figure 4 we see that it is $m = \frac{1}{2}$. So we now know that the equation of the line is $y = \frac{1}{2}x + c$, where we still have to find c. The best way to do this is to use algebra rather than geometry. Of course we could get a piece of graph paper, draw the line and see where the line cuts the y-axis but not only would this be a lot of work, it would only give us an approximation for c. So let us use algebra. We know

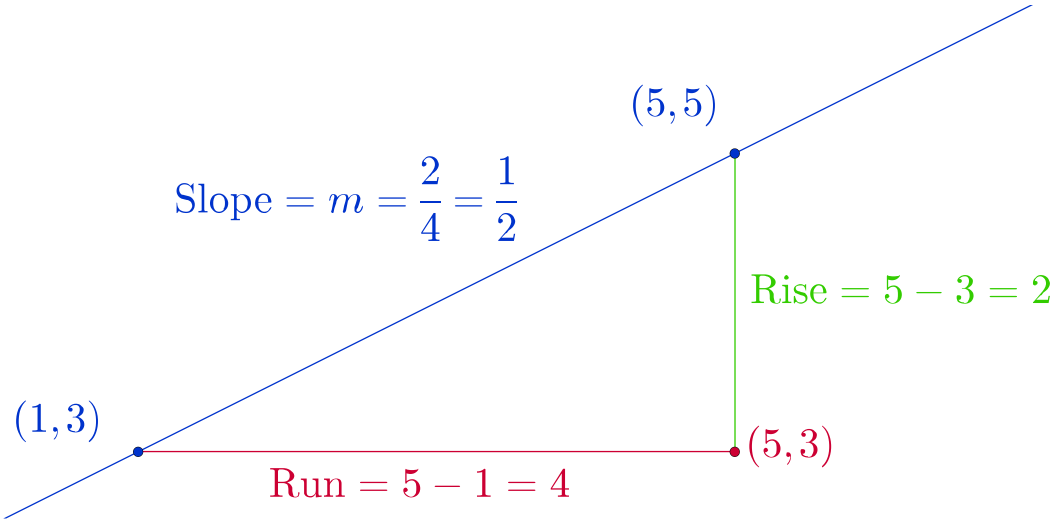


FIGURE 4. Finding the slope of a line given two points on it.

that the line passes through $(1, 3)$, so putting $x = 1$ and $y = 3$ into $y = \frac{1}{2}x + c$ will allow us to find c. We get $3 = \frac{1}{2} \times 1 + c$, that is $3 = \frac{1}{2} + c$. Subtracting $\frac{1}{2}$ from each side of this equation we obtain $c = \frac{5}{2}$. Thus the equation of the line is $y = \frac{1}{2}x + \frac{5}{2}$.
Note that we could equally well use $(5, 5)$ to find c and of course, we **MUST** get the same answer; if we don't then we must have made a mistake somewhere.
Once we have obtained the equation, then a good check on our working is to check the two points we started with do indeed lie on $y = \frac{1}{2}x + \frac{5}{2}$. For example, if we put $x = 1$ into $y = \frac{1}{2}x + \frac{5}{2}$, we get $y = \frac{1}{2}(1) + \frac{5}{2} = \frac{1}{2} + \frac{5}{2} = 3$, so $(1, 3)$ does indeed lie on the line.

Now for an example where the slope of the line is negative, that is, it slopes down as we go from left to right.

Example 2.1.3. Find the equation of the line through the points $(-1, 4)$ and $(2, -2)$.
Again we will first find the slope of the line.

Looking at Figure 5 we see that it is $m = -2$. We now find c by using the fact that the point $(-1, 4)$ lies on the line. Substituting $x = -1$ and $y = 4$ into $y = -2x + c$ we get $4 = -2(-1) + c$, that is $4 = 2 + c$. Hence (subtracting 2 from both sides of the equation) $c = 2$ and so the equation of the line is $y = -2x + 2$.
Again, it is now a good idea to check that both the points we started with do lie on $y = -2x + 2$.

Now let us do an example where the line is vertical.

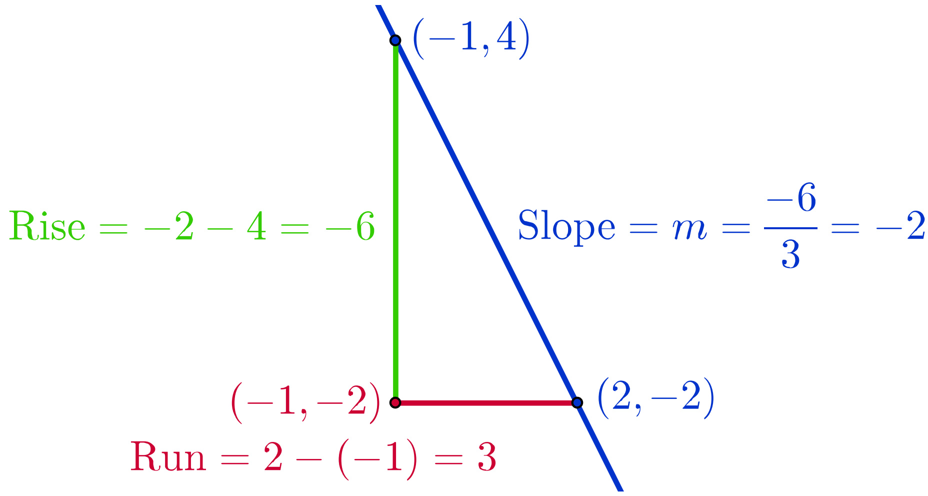


FIGURE 5. Finding the slope of a line given two points on it.

Example 2.1.4. Find the equation of the line through the points $(-5, 4)$ and $(-5, 3)$.
Here we notice that the x-coordinate of each of the points is the same (i.e., -5), so the line must be vertical. Hence its equation is $x = -5$.

We can in fact generalise the technique we used in Example 2.1.2 and Example 2.1.3 to any non-vertical line and this is what we will do next.

Example 2.1.5. Find the equation of the line through the points (x_1, y_1) and (x_2, y_2) given that $x_1 \neq x_2$.
Since $x_1 \neq x_2$ the line is not vertical and we can use the same method as in Example 2.1.2 and Example 2.1.3. If $x_1 = x_2$, then we couldn't use this method, since we wouldn't be able to divide by $x_2 - x_1$ because it would be zero.

From Figure 6 we see that the slope of the line is $\frac{y_2 - y_1}{x_2 - x_1}$. Note that the order of the x_1 and x_2 and the y_1 and the y_2 don't matter provided that they match. That is $\frac{y_1 - y_2}{x_1 - x_2}$ would also be correct but $\frac{y_2 - y_1}{x_1 - x_2}$ or $\frac{y_1 - y_2}{x_2 - x_1}$ would give the negative of the correct slope.

We now know that the equation of the line is

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + c, \tag{2}$$

where we still have to find c. As in Example 2.1.2 and Example 2.1.3 we do this by substituting (2) one of the points that we were given at the start of the example. If

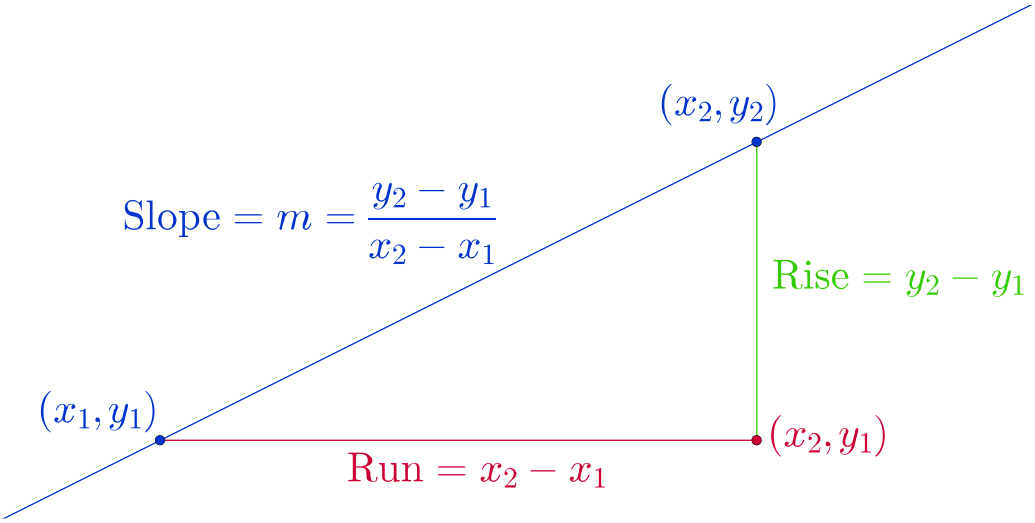


FIGURE 6. Finding the slope of a line given two points on it.

we use (x_1, y_1) then we obtain $y_1 = \frac{y_2 - y_1}{x_2 - x_1}x_1 + c$, so

$$\begin{aligned} c &= y_1 - \frac{y_2 - y_1}{x_2 - x_1}x_1 \\ &= \frac{y_1(x_2 - x_1) - (y_2 - y_1)x_1}{x_2 - x_1} \\ &= \frac{y_1x_2 - y_1x_1 - y_2x_1 + y_1x_1}{x_2 - x_1} \\ &= \frac{y_1x_2 - y_2x_1}{x_2 - x_1}. \end{aligned}$$

Note that if we use the other point (x_2, y_2) then we get exactly the same expression for c. We can now substitute for c in (2) to obtain

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + \frac{y_1x_2 - y_2x_1}{x_2 - x_1} \tag{3}$$

which is the equation of the line.

Warning 2.1.6. One method of finding the equation of a line given two points on it would be to memorize (3) and use it whenever you need it. I **DON'T** recommend this approach. It is far better to understand the derivation of (3) and be able to do it yourself whenever you need. In this way you will never have to worry if you have remembered it correctly, or should the x_1 and the x_2 be in the opposite order etc.

Another sort of problem is that we might be asked to find the equation of a line through a point in a particular direction. Here are some examples of this.

Example 2.1.7. Find the equation of the line through the point $(-2, 4)$ parallel to the line $y = 3x + 1$.
Here we know the line is parallel to the line $y = 3x + 1$, so it must have the same

slope, i.e., $m = 3$. So we already know its equation is $y = 3x + c$. To find c, we let $y = 4$ and $x = -2$. Thus $4 = 3(-2) + c$ which yields $c = 4 + 6 = 10$. Hence the equation of the line is $y = 3x + 10$.

Example 2.1.8. Find the equation of the line through the point $(4, 0)$ parallel to the line through the points $(0, 1)$ and $(2, -5)$.
Here we know the line is parallel to the line through the points $(0, 1)$ and $(2, -5)$, so to find the slope of the line we want, we find the slope of the line through the points $(0, 1)$ and $(2, -5)$. This is

$$\frac{-5 - 1}{2 - 0} = \frac{-6}{2} = -3.$$

So the equation of the required line is $y = -3x+c$. To find c, we let $y = 0$ and $x = 4$. Thus $0 = -3(4) + c$, so $c = 12$. Hence the equation of the line is $y = -3x + 12$.

Remark 2.1.9. Sometimes instead of being given a point on the line we might be given the value of c (as well as the direction of the line) but this is no problem, it just makes the question easier.

2.2. **Sketching Lines**.

Another problem we might be faced with is to sketch a line given its equation, so in this section we will do a couple of examples of this.

Example 2.2.1. Sketch the graph of the line with equation $y = 4x + 5$.
The easiest way to approach this problem is to pick a couple of values for x, calculate the corresponding y values and then draw a line through the two points we have found. We will first take $x = 0$ which gives $y = 5$, so we have the point $(0, 5)$. Note there is a general feature of doing maths here; if we have a choice as to which value we can pick, then we may as well pick the one that makes the calculation easiest! For the other point, we could pick $x = 1$, which gives the next easiest calculation. This might be fine depending on the scale of the graph we want to sketch. The main thing to bear in mind is that the two points we pick should be a reasonable distance away from each other on the page, for otherwise the accuracy of our sketch will be poor. In this case, say we want to concentrate on the graph between the points $x = 0$ and $x = 10$ then $x = 10$ would be a good choice for our other point (the calculation is still very easy with $x = 10$). This gives $y = 45$, so our second point is $(10, 45)$. We can now sketch the graph which is shown in Figure 7.

Remark 2.2.2. Note that when sketching a graph, the scales of the x and y axes don't have to be the same. In this case, to get a nice coverage of the page, I have taken the same distance to represent one unit on the x-axis and ten units on the y-axis.

Also note it is good practice to label the graph. This is a general feature of writing mathematics. It is not only important to give the correct answer, it is also important to include words to explain what your answer is.

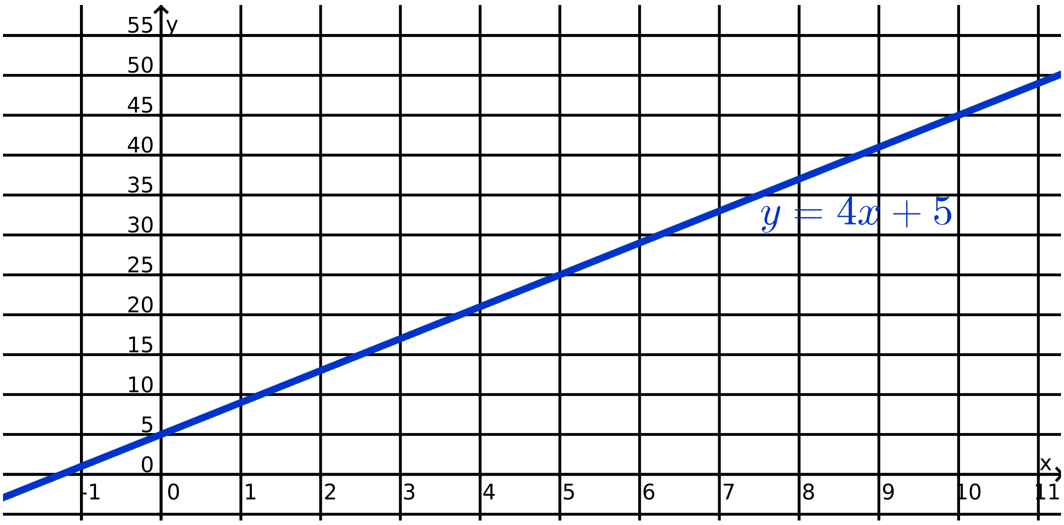


FIGURE 7. Graph of the line $y = 4x + 5$.

Example 2.2.3. Sketch the graph of the line with equation $y = -2x - 1$ concentrating on the region between $x = -5$ and $x = 5$.
Here we want to concentrate on the region between $x = -5$ and $x = 5$, so we will take these as our two x values. This will be a little more work than taking $x = 0$ and $x = 1$, say, but we will get a more accurate graph. When $x = -5$, $y = 9$ and when $x = 5$, $y = -11$, so our two points are $(-5, 9)$ and $(5, -11)$. This gives the graph shown in Figure 8.

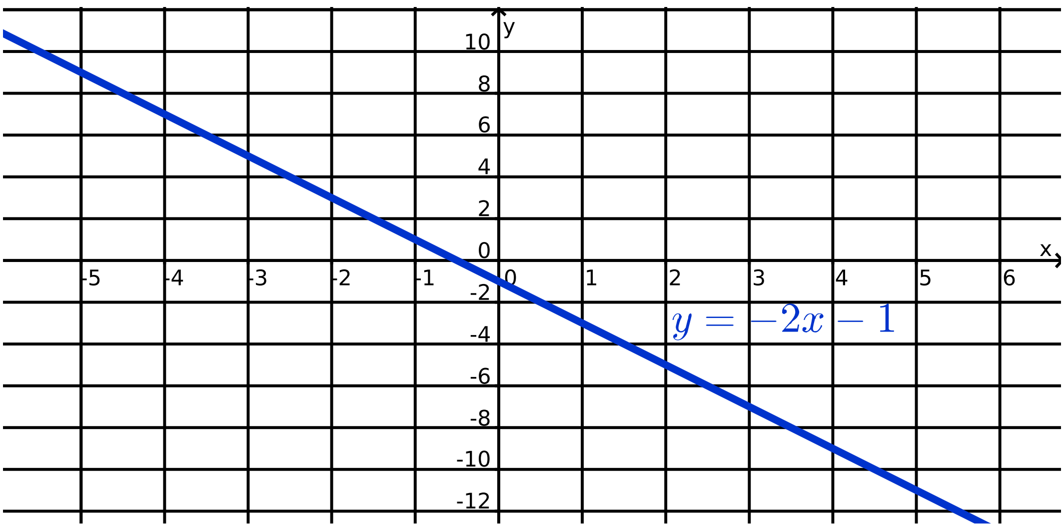


FIGURE 8. Graph of the line $y = -2x - 1$.

Remark 2.2.4. Here I have chosen the same distance to represent one unit on the x-axis and two units on the y-axis to get a nice coverage of the page.

Also note that sometimes the equation of the line will not be given in the form $y = mx+c$. For example $2y+4x = -2$ also represents the line in Figure 8. However this does not present any great problem, since we can always rearrange the equation to be in the form $y = mx + c$ (except for vertical lines).

I have also prepared a GeoGebra worksheet that will allow you to change the values of m and c and see what effect this has on the line. It can be found at http://www.ucd.ie/msc/access/straightlinegraph/. I would recommend that you have a play around with this worksheet since it makes it much easier to see what happens when you can see the graph changing as you move the slider. Note that you can reset the graph to its starting position by clicking on the icon in the top right hand corner of the worksheet.

2.3. **Solving Linear Equations**.

So far we have looked at finding the equation of lines and sketching their graphs. In this section we will examine the connection between on one hand, equations and graphs of lines and on the other, solving linear equations.

Let us start with a definition.

Definition 2.3.1. A *linear* equation is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

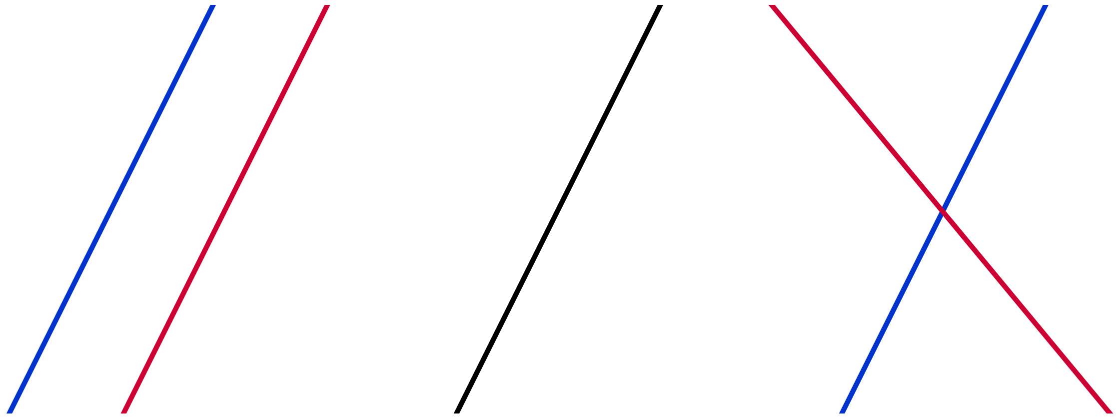
where $a_1, a_2, \ldots, a_n$ and b are constants and $x_1, x_2, \ldots, x_n$ are unknowns.

Remark 2.3.2. Definition 2.3.1 covers the general case but in this course, we will not cover the case where we have more than two unknowns, so the most complicated form of linear equation that we will study is $a_1x_1 + a_2x_2 = b$. Note that in this case we will often call the unknowns x and y rather than x_1 and x_2.

Before we proceed any further, I want to point out that when we use an equation to describe a line and when we solve a linear equation, we are using algebra in two completely different ways. An equation describing a line is a *rule* that tells us what the y value is for any given x value. On the other hand, when we solve equations, we are finding *unknown* values.

The simplest form of linear equation is one where we have only one variable; for example $3x + 2 = 6$ is such an equation. To solve this equation we subtract 2 from each side to get $3x = 4$ and then divide both sides by 3 to obtain the solution $x = \frac{4}{3}$.

The next simplest form of linear equation is one where we have two variables; for example $2x + y = 4$. In this case we might ask ourselves, what does it mean to find a solution? The answer is that a solution is any two numbers x and y such that $2x + y = 4$. For example $x = 0$ and $y = 4$ is a solution and $x = 1$ and $y = 2$ is another one. This presents us with a bit of a problem, since how can we tell how many solutions there are and how do we know when we have found them all? One way is to rearrange the equation to obtain $y = -2x + 4$, so we know that if we take



(A) Parallel lines (B) Lines the same (C) Intersect at a point

FIGURE 9. Different ways lines can intersect

any real number x and then take $y = -2x + 4$, then the pair $\{x, y\}$ is a solution and these are the only possible solutions (there are infinitely many of them).

Another approach we can take is to spot that $y = -2x + 4$ is the equation of a line with slope -2 which cuts the y-axis at $(0, 4)$. This means is that any pair $\{x, y\}$ such that (x, y) lies on this line is a solution. What we have done here is something that is very often done in Mathematics. We have converted an algebraic problem (involving equations) into a geometric one (involving lines). This can be useful since in some cases it might be easier to solve a geometric problem than an algebraic one. Of course the opposite can also be true; sometimes it might be easier to solve an algebraic problem and in this case we will try to convert the problem in the opposite direction. In other cases, the problem may still be as difficult but expressing the problem in another way may give us a deeper insight into it, which is always a good thing.

2.4. **Solving Simultaneous Linear Equations**.

While it is easy to find solutions to a linear equation with two variables, things get a bit more complicated if we want to simultaneously find solutions to two linear equations with two variables. Here again we can attack the problem using geometry or algebra. Our approach will be first to get an overview of the possible types of solutions using geometry but then use algebra to solve particular problems (using geometry we can only get approximate solutions).

So, using a geometrical argument, let us think about what sort of solutions are possible. As we noted above, the solutions to a linear equation may be regarded as the points lying on a line. So, finding solutions to two linear equations in two variables is equivalent to finding points that lie on two lines at the same time. However, given two lines, there are three essentially different possibilities as to how they meet. An example of each is shown in Figure 9.

The first possibility is shown in Figure 9A. Here the lines are parallel but not equal, so they never meet. So in this case the simultaneous equations have no solutions.

The second possibility is shown in Figure 9B. Here the lines are equal, so every point on the line is a solution. Hence in this case, the simultaneous equations have infinitely many solutions.

The last possibility is shown in Figure 9C. Here the line are not parallel so they meet in exactly one point. So in this case the simultaneous equations have exactly one solution.

Remark 2.4.1. While we won't study it in this course, I think it is still interesting to point out that no matter how many linear equations we have, in no matter how many variables, there are still only these three possibilities. That is, there are no solutions, one solution or infinitely many solutions. So, for example, it is not possible to have two solutions to any set of simultaneous linear equations.

Now that we have used geometry to get an overview of the possibilities, let us solve some actual problems using algebra.

Example 2.4.2. Solve the simultaneous equations

$$-3x + y = -1 \tag{4}$$

$$5x + 2y = 20 \tag{5}$$

There are essentially two different ways to solve simultaneous equations (the exact procedure may vary slightly in some exceptional cases, for example, if the coefficient of one of the variables is zero in one of the equations). In the first method we can use one equation to express one of the variables, x say, as a function of y and substitute this into the second equation to give an equation that can then be solved for y. We then substitute this value of y into either equation to find x. In the other method we multiply one or both of the equations by a non-zero number so that when we add or subtract the equations one of the variables will cancel and we can then solve the resulting equation for the other variable. As in the first method, we then use either of the equations to find the other variable.

Let us solve these equations by using each method in turn.

First method: Here I will rearrange (4) to obtain $y = 3x - 1$. (note I could also have used (5) but since the coefficients of x and y in (5) are 5 and 2, respectively, this would involve introducing fractions and it is wise to avoid this if possible). We now let $y = 3x - 1$ in (5) to obtain $5x + 2(3x - 1) = 20$. Then

$$5x + 6x - 2 = 20 \Rightarrow 11x = 22 \Rightarrow x = 2.$$

We can now substitute $x = 2$ into $y = 3x - 1$ (equivalent to the first equation) to obtain $y = 3(2) - 1 = 5$. Hence the solution to the set of simultaneous equations is $x = 2$ and $y = 5$. So in this case we are in the situation shown in Figure 9C, where there is a unique solution.

At this stage, a very good check on our working is to substitute the values of x and y back into the original equations to make sure that they do indeed satisfy them.

This doesn't mean that we definitely have a totally correct solution (for example, potentially there could be infinitely many solutions rather than one) but it does rule out most potential mistakes.

Now let us solve the problem using the second method. There are several ways to multiple the equations to make one of the variables cancel. Perhaps the easiest is to multiply (4) by 2 and then subtract (5) from it (this will make the y's cancel).

$$\begin{array}{rrcrcr} & -6x & + & 2y & = & -2 \\ - & 5x & + & 2y & = & 20 \\ \hline & -11x & & & = & -22 \end{array}$$

So $-11x = -22$ and hence $x = 2$. We can now proceed as we did in the first method to again obtain $x = 2$ and $y = 5$ as the solution.

Example 2.4.3. Solve the simultaneous equations

$$4x + 3y = -6 \tag{6}$$

$$-3x + 2y = 13 \tag{7}$$

Again we will solve the problem by two methods.

Method 1: Here no matter what we do we will end up with a fraction, so it doesn't really matter which equation we choose. Using (6), we obtain $4x = -3y - 6$ so that $x = -\frac{3}{4}y - \frac{3}{2}$. On substituting this into (7) we get

$$-3\left(-\frac{3}{4}y - \frac{3}{2}\right) + 2y = 13 \Rightarrow \frac{9}{4}y + \frac{9}{2} + 2y = 13 \Rightarrow \frac{17}{4}y = \frac{17}{2} \Rightarrow y = 2.$$

Substituting $y = 2$ into (6) we get $4x + 3(2) = -6$, so $4x = -12$ and $x = -3$. Thus the solution is $x = -3$ and $y = 2$; again we are in the situation shown in Figure 9C.

Method 2: It is usually simpler if we work with integer multiples of the equations, so in this case, we will add four times (7) to three times (6) (this will eliminate the x's).

$$\begin{array}{rrcrcr} & 12x & + & 9y & = & -18 \\ + & -12x & + & 8y & = & 52 \\ \hline & & & 17y & = & 34 \end{array}$$

Hence $17y = 34$, so we again obtain $y = 2$. To complete the solution we now proceed as in the first method.

Example 2.4.4. Solve the simultaneous equations

$$2x + 5y = -2 \tag{8}$$

$$-4x - 10y = 4 \tag{9}$$

Let us try to use the first method to solve this problem and see what happens. From (8), we obtain $2x = -5y - 2$, so that $x = -\frac{5}{2}y - 1$. If we substitute this into (9), we get

$$-4\left(-\frac{5}{2}y - 1\right) - 10y = 4 \Rightarrow 0 = 0.$$

Of course it is true that $0 = 0$ but where does that leave us if we are trying to solve the problem. The key is to spot that (9) is minus two times (8), so they are equivalent equations (they both represent the same line). Thus we are in the situation shown in Figure 9B and there are infinitely many solutions. The solutions are $x = -\frac{5}{2}y - 1$, where y is any real number. We won't worry about it too much in this course but when there are infinitely many solutions, we write them in terms of a *parameter*. In this case we could write the solutions as $x = -\frac{5}{2}t - 1$ and $y = t$, where t is a real number. In this case t is called a *free variable*; we will return to free variables in the second semester.

Note there is no reason why we have to let $y = t$, we could also let $x = t$ and in this case the solutions would be written as $x = t$ and $y = -\frac{2}{5}t - \frac{2}{5}$ (where this expression for y is obtained from either (8) or (9)).

Also note that if we attempt the second method in a case like this, then we will also obtain $0 = 0$ after adding twice (8) to (9).

Example 2.4.5. Solve the simultaneous equations

$$2x - 3y = 5 \tag{10}$$

$$-4x + 6y = 10 \tag{11}$$

Again let us try to solve this problem using the first method. From (10) we obtain $2x = 3y + 5$, so $x = \frac{3}{2}y + \frac{5}{2}$. When we substitute this into (11) we obtain

$$-4\left(\frac{3}{2}y + \frac{5}{2}\right) + 6y = 10 \Rightarrow -6y - 10 + 6y = 10 \Rightarrow -10 = 10.$$

It is certainly not true that $-10 = 10$, so what has gone wrong? The explanation is that (10) and (11) represent parallel lines, so we are in the situation shown in Figure 9A and there is no simultaneous solution to (10) and (11). This will always be the case if we end up with something that is false like $-10 = 10$, unless we have made a mistake!

Note that if we add two times (10) to (11) we end up with $0 = 20$, so again we know the equations have no simultaneous solution, provided we have not made a mistake.

I have also prepared a GeoGebra worksheet that will allow you to solve simultaneous equations by entering the equations into the worksheet. It can be found at http://www.ucd.ie/msc/access/simultaneousequations/. Please have a play around with this worksheet but do also practice solving equations by hand, since this is what you will have to do in the exam. Also note that in the assignment questions, you have to show your working to obtain the marks. As with the straight line graph, you can always reset the worksheet to its starting position by clicking on the icon in the top right hand corner of the worksheet, so please don't worry that you can break it in any way.

2.5. Pythagoras and the Length of a Line Segment .

Another thing we might want to do is to find the length of a line segment between two points. Recall that Pythagoras' Theorem says that in a right angled triangle, the length of the hypotenuse (the side opposite the right angle) squared is equal to the sum of the squares of the lengths of the other two sides. In Figure 10 we will show how this theorem can be used to find the length of a line segment.

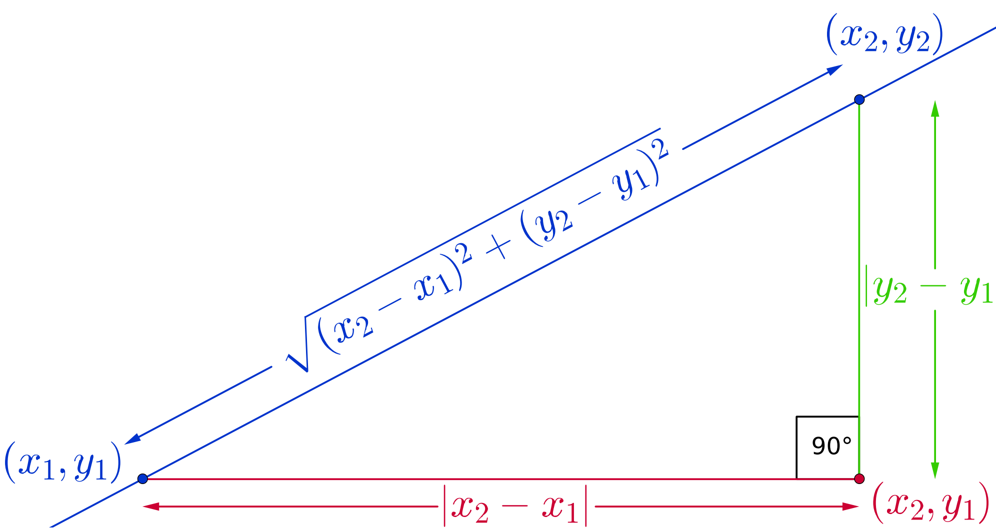


FIGURE 10. Length of a line segment.

Say we want to find the length of the line segment between the points (x_1, y_1) and (x_2, y_2). Then the key is to form a right angled triangle where two of the sides are parallel to the x and y axes and the hypotenuse is the line segment that we want to calculate the length of. Since the lengths of the sides parallel to the x and y axes are $|x_2 - x_1|$ and $|y_2 - y_1|$, respectively, it follows from Pythagoras' Theorem that the length of the line segment is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Remark 2.5.1. Depending on the actual values of (x_1, y_1) and (x_2, y_2), it could be that the lengths of the sides parallel to the x and y axes are $x_1 - x_2$ or $y_1 - y_2$ rather than $x_2 - x_1$ or $y_2 - y_1$, so we have to use absolute value signs to cover all eventualities. We don't need them in $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ however since $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$.

Also note that if $x_2 - x_1 = 0$ or $y_2 - y_1 = 0$ then we don't actually have a triangle but the formula still works. For example, if $x_2 - x_1 = 0$, then the line segment is parallel to the y-axis and its length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |y_2 - y_1|$ as we want. Similarly if $y_2 - y_1 = 0$, then the line segment is parallel to the x-axis and its length is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |x_2 - x_1|$.

Here are some examples.

Example 2.5.2. Find the length of the line segment between $(-1,-2)$ and $(3,2)$. Here we will let $(x_1,y_1)=(-1,-2)$ and $(x_2,y_2)=(3,2)$. Hence its length is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{(3-(-1))^2+(2-(-2))^2}\\ &= \sqrt{4^2+4^2}\\ &= \sqrt{16+16}\\ &= \sqrt{32}\\ &= 4\sqrt{2}.\end{aligned}$$

Note that we could equally well let $(x_1,y_1)=(3,2)$ and $(x_2,y_2)=(-1,-2)$, our final answer would be the same.

Example 2.5.3. Find the length of the line segment between $(3,6)$ and $(-2,3)$. Here we will let $(x_1,y_1)=(3,6)$ and $(x_2,y_2)=(-2,3)$. Hence its length is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{(-2-3)^2+(3-6)^2}\\ &= \sqrt{(-5)^2+(-3)^2}\\ &= \sqrt{25+9}\\ &= \sqrt{34}.\end{aligned}$$

Example 2.5.4. Find the length of the line segment between $(3,5)$ and $(3,4)$. Here we will let $(x_1,y_1)=(3,5)$ and $(x_2,y_2)=(3,4)$. Hence its length is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{(3-3)^2+(4-5)^2}\\ &= \sqrt{0^2+(-1)^2}\\ &= \sqrt{0+1}\\ &= \sqrt{1}\\ &= 1.\end{aligned}$$

Note this line segment is parallel to the y-axis.

2.6. **Midpoint of a Line Segment**.

For our final section of this chapter, we will look at how to find the midpoint of a line segment.

First note that a point B is the midpoint of a line segment AC if the distance from B to A is the same as the distance from B to C (this is just what the word 'midpoint' means).

Our aim is to show that the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ is the midpoint of the line joining (x_1,y_1) to (x_2,y_2). In order to do this, let us examine Figure 11.

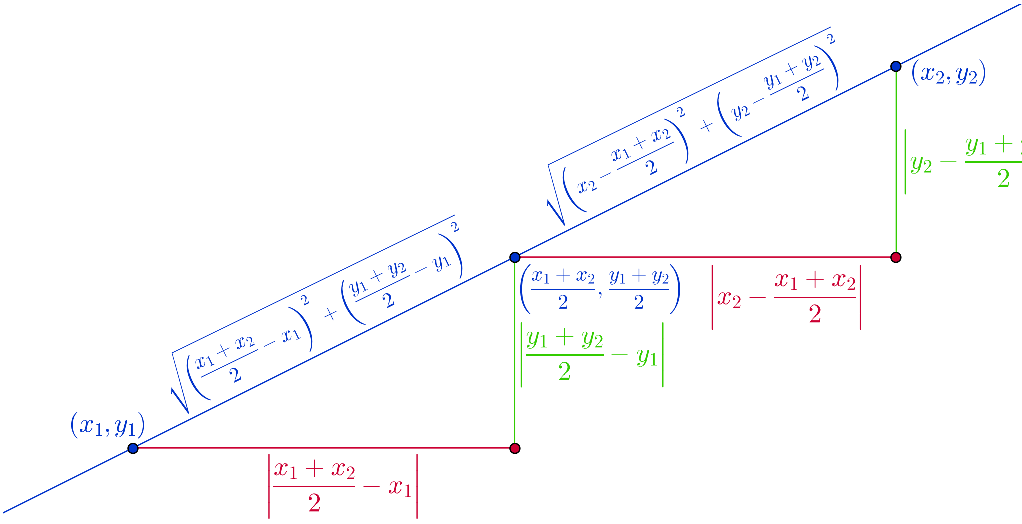


FIGURE 11. Midpoint of a line segment.

Using the formula for the length of a line segment that we derived in Section 2.5, it follows that the distance from $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ to (x_1, y_1) is

$$\sqrt{\left(\frac{x_1+x_2}{2}-x_1\right)^2+\left(\frac{y_1+y_2}{2}-y_1\right)^2} \tag{12}$$

and that the distance from $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ to (x_2, y_2) is

$$\sqrt{\left(x_2-\frac{x_1+x_2}{2}\right)^2+\left(y_2-\frac{y_1+y_2}{2}\right)^2}. \tag{13}$$

So, to prove our claim, we have to show that the expressions in (12) and (13) are equal.

However

$$\sqrt{\left(\frac{x_1+x_2}{2}-x_1\right)^2+\left(\frac{y_1+y_2}{2}-y_1\right)^2}$$

$$\begin{aligned}
&= \sqrt{\left(\frac{x_1+x_2-2x_1}{2}\right)^2+\left(\frac{y_1+y_2-2y_1}{2}\right)^2}\\
&= \sqrt{\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2-y_1}{2}\right)^2}\\
&= \sqrt{\left(\frac{2x_2-(x_1+x_2)}{2}\right)^2+\left(\frac{2y_2-(y_1+y_2)}{2}\right)^2}\\
&= \sqrt{\left(x_2-\frac{x_1+x_2}{2}\right)^2+\left(y_2-\frac{y_1+y_2}{2}\right)^2}.
\end{aligned}$$

Let us finish by doing a couple of examples.

Example 2.6.1. Find the midpoint of the line segment joining $(2,-4)$ and $(-3,7)$. If we let $(x_1,y_1)=(2,-4)$ and $(x_2,y_2)=(-3,7)$, then using the formula, we have that the midpoint is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{2+(-3)}{2},\frac{-4+7}{2}\right)=\left(-\frac{1}{2},\frac{3}{2}\right).$$

Remark 2.6.2. As was the case with the length of a line segment, it doesn't matter which point we take to be (x_1,y_1) and which point we take to be (x_2,y_2).

Example 2.6.3. Find the midpoint of the line segment joining $(1,0)$ and $(1,-5)$. If we let $(x_1,y_1)=(1,0)$ and $(x_2,y_2)=(1,-5)$, then using the formula, we have that the midpoint is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{1+1}{2},\frac{0+(-5)}{2}\right)=\left(\frac{2}{2},\frac{-5}{2}\right)=\left(1,-\frac{5}{2}\right).$$